ON THE ASSOCIATIVITY OF GLUING

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ABSTRACT. This paper studies the associativity of gluing of trajectories in Morse theory. We show that the associativity of gluing follows from of the existence of compatible manifold with face structures on the compactified moduli spaces. Using our previous work, we obtain the associativity of gluing in certain cases.

In particular, associativity holds when the ambient manifold is compact and the vector field is Morse-Smale.

1. INTRODUCTION

In order to develop his homology theory, Floer invented two techniques in Morse theory (see e.g. [15]). One is the compactification of the moduli spaces of negative gradient trajectories. The other one is the gluing of broken trajectories. These two arguments have continuously impacted Morse theory since then. For example, moduli spaces have extensive applications in geometry and topology (see e.g. [16], [18], [1]-[3] and [5]-[13]).

Due to this influence, there is a folklore theorem or rather a philosophy as follows. Under certain conditions of compactness, a moduli space of trajectories can be compactified to be a manifold with corners. There has been some progress on this topic in the literature as it was interpreted and proved in certain cases. For example, see [18, Proposition 2.11], [3, Theorem 1], [2, Appendix], [20, Theorem 3.3] and [21, Theorem 7.5].

Another related problem is the so-called "associativity of gluing" that is alluded to in the title. We first learned of this problem in the paper of Cohen, Jones and Segal [7].

This paper shows that the associativity of gluing is a direct consequence of the existence of compatible manifold structures on the compactified moduli spaces. We will in fact see that there is a general result along these lines in which Morse theory occurs as a special case.

Suppose p_1, p_2 and p_3 are critical points, γ_1 is a trajectory from p_1 to p_2 and γ_2 is a trajectory from p_2 to p_3. In a strict sense, the pair (γ_1, γ_2) of consecutive trajectories is not a trajectory. We consider (γ_1, γ_2) as a broken trajectory from p_1 to p_3. A *gluing* of (γ_1, γ_2) is a smooth family

$$\gamma_1 \#_\lambda \gamma_2 ,$$

where $\lambda \in [0, \epsilon)$ is the gluing parameter, $\gamma_1 \#_0 \gamma_2 = (\gamma_1, \gamma_2)$ and $\gamma_1 \#_\lambda \gamma_2$ is an unbroken trajectory when $\lambda \neq 0$.

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Suppose now that γ_1, γ_2 and γ_3 are three consecutive trajectories. Then one can form two families according to the various ways of associating pairs:

$$(\gamma_1 \#_{\lambda_1} \gamma_2) \#_{\lambda_2} \gamma_3 \qquad \text{and} \qquad \gamma_1 \#_{\lambda_1} (\gamma_2 \#_{\lambda_2} \gamma_3)\,.$$

If these families coincide, one says that associativity gluing is satisfied.

The manifold structure of a compactified moduli space is actually related to the associativity of gluing. One can derive the manifold structure from the associativity of gluing because the latter provides nice coordinate charts for the former. However, this is *not* the only way to get the manifold structure. The papers [18], [3], [20] and [21] do not use any gluing arguments.

In this paper, we shall strengthen the above relationship by working in the opposite direction: we will show that the associativity of gluing is a consequence of the existence of a certain kind of manifold structure. More precisely, Theorems 3.2 and 3.3 show that, if the manifold structures satisfy Assumption 3.1, then one will get the associativity of gluing for free. In fact, we reformulate a gluing of broken trajectories as parametrizations of collar neighborhoods of the strata of the compactified moduli spaces. Then associativity of gluing will be seen to be equivalent to a choice of compatible collar structure. The above theorems will be generalized to Theorem 4.4 which is a statement about the compatible collar structures of manifold with faces.

In short, these theorems convert the problem of the associativity of gluing to the problem of manifold structures. By the results we proved about manifold structures in [20] and [21], we get Propositions 3.4 and 3.5. They show the associativity of gluing in Morse theory in two contexts. An informal restatement of these results is given by

Corollary A. *Suppose M is a compact Riemannian manifold and f is a Morse function on M. Suppose $-\nabla f$ satisfies the Morse-Smale condition.*

Then there exists an associative gluing rule.

Corollary B. *Suppose M is a complete Hilbert-Riemannian manifold. Assume f satisfies Condition (C) and has finite indices. Suppose $-\nabla f$ satisfies the Morse-Smale condition. Assume that the metric on M is locally trivial (see* [20, Definition 2.16]*).*

Then there exists an associative gluing rule.

A byproduct of our work is Proposition 7.1 which is also about compatible collar structures. Theorem 4.4 is about a family of manifolds with faces (see Assumption 4.1), while Proposition 7.1 is about a single one. However, the assumption of Proposition 7.1 is more general.

The outline of this paper is as follows. Section 2 reviews the definition of moduli spaces of trajectories. Section 3 gives our main results on the associativity of gluing. Section 4 generalizes the theorems in the previous section. The proof of our main theorem occupies Sections 5 and 6. We conclude this paper by presenting the byproduct in Section 7.

2. Moduli Spaces

In this section, we review the definition of the moduli spaces of trajectories of negative gradient vector fields. (See [22] or [20] for more details.)

Suppose M is a Hilbert-Riemannian manifold and f is a Morse function on M. Let $-\nabla f$ be the negative gradient of f.

Definition 2.1. Let $\phi_t(x)$ be the flow generated by $-\nabla f$ with initial value x. Suppose p is a critical point. Define the descending manifold of p as $\mathcal{D}(p) = \{x \in M \mid \lim_{t\to-\infty} \phi_t(x) = p\}$. Define the ascending manifold of p as $\mathcal{A}(p) = \{x \in M \mid \lim_{t\to+\infty} \phi_t(x) = p\}$.

Both $\mathcal{D}(p)$ and $\mathcal{A}(p)$ are smoothly embedded submanifolds in M.

Definition 2.2. If the descending manifold $\mathcal{D}(p)$ and the ascending manifold $\mathcal{A}(q)$ are transversal for all critical points p and q, then we say $-\nabla f$ satisfies the transversality or Morse-Smale condition.

If $-\nabla f$ satisfies transversality, then $\mathcal{D}(p) \cap \mathcal{A}(q)$ is an embedded submanifold which consists of points on trajectories (or flow lines) from p to q. Since a trajectory has an $\mathbb{R}$-action, we may take the quotient of $\mathcal{D}(p) \cap \mathcal{A}(q)$ by this $\mathbb{R}$-action, i.e. consider its orbit space acted upon by the flow. This leads to the following definition.

Definition 2.3. Suppose $-\nabla f$ satisfies transversality. Define $\mathcal{W}(p,q) = \mathcal{D}(p) \cap \mathcal{A}(q)$. Define the moduli space $\mathcal{M}(p,q)$ as the orbit space $\mathcal{W}(p,q)/\mathbb{R}$.

We assume transversality all through this paper. It's well known that, when f has finite indices, $\mathcal{M}(p,q)$ is a finitely dimensional manifold of dimension $\operatorname{ind}(p) - \operatorname{ind}(q) - 1$, where $\operatorname{ind}(*)$ is the Morse index of $*$.

Definition 2.4. Suppose p and q are two critical points. We define the relation $p \succeq q$ if there is a trajectory from p to q. We define the relation $p \succ q$ if $p \succeq q$ and $p \neq q$.

The transversality implies that " $\succeq$ " is a partial order. To guarantee this, it suffices to show the transitivity of " $\succeq$ ". The best proof is probably to use the λ-Lemma (see [19, p. 85, Corollary 1]). It is valid even if M is a Banach manifold and the vector field is a general one (*not* necessarily a negative gradient) with hyperbolic singularities. In Floer theory (see e.g. [15, p. 529]), this can be proved by a gluing argument.

Definition 2.5. An ordered set $I = \{r_0, r_1, \cdots, r_{k+1}\}$ is a critical sequence if r_i $(i = 0, \cdots, k+1)$ are critical points and $r_0 \succ r_1 \succ \cdots \succ r_{k+1}$. We call r_0 the head of I, and r_{k+1} the tail of I. The length of I is $|I| = k$.

Suppose $I = \{r_0, r_1, \cdots, r_{k+1}\}$ is a critical sequence. We define the following product manifold

$$\mathcal{M}_I = \prod_{i=0}^{k} \mathcal{M}(r_i, r_{i+1}). \tag{2.1}$$

Each element in $\mathcal{M}_I$ stands for a (un)broken trajectory from r_0 to r_{k+1} which is broken at exactly the points r_i $(i = 1, \cdots, k)$.

3. Main Theorems

In this section, we state our results on the associativity of gluing.

Theorems 3.2 and 3.3 will be based on the following assumption. For the definitions of manifold with faces and the k-stratum, see Definitions 4.3 and 4.2.

Assumption 3.1. *Suppose Ω is the set of critical points of f. Assume Ω is countable. The relation "$\succeq$" (see Definition 2.4) defined on Ω is a partial order. Suppose $\mathcal{M}(p,q)$ is a finite dimensional manifold for each $p,q\in\Omega$ such that $p\succ q$ (see Remark 3.1). Suppose $\mathcal{M}(p,q)$ can be compactified to $\overline{\mathcal{M}(p,q)}$ having the structure of a compact smooth manifold with faces. In addition, assume each $\overline{\mathcal{M}(p,q)}$ satisfies the following conditions:*

(1). We have $\overline{\mathcal{M}(p,q)}=\bigsqcup_I \mathcal{M}_I$, where the disjoint union is over all critical sequences with head p and tail q. The k-stratum of $\overline{\mathcal{M}(p,q)}$ is $\bigsqcup_{|I|=k}\mathcal{M}_I$, and each M_I is an open subset of the k-stratum. The smooth structure of $\overline{\mathcal{M}(p,q)}$ is compatible with those of $\mathcal{M}_I$.

(2). Suppose $p\succ r\succ q$, then the natural inclusion $\overline{\mathcal{M}(p,r)}\times\overline{\mathcal{M}(r,q)}\hookrightarrow\overline{\mathcal{M}(p,q)}$ is a smooth embedding.

Remark 3.1. By Definition 2.3, $\mathcal{M}(p,q)$ has a natural smooth structure induced from those of $\mathcal{D}(p)$ and $\mathcal{A}(q)$ (see e.g. [22], [18], [3], [20] and [21]). However, in order to make Assumption 3.1 hold, we may give $\mathcal{M}(p,q)$ a smooth structure different from the above one (see Remark 3.3).

In order to make the statement of gluing conceptual and strong, we shall have to introduce the following formal definitions.

Suppose $I_1=\{r_0,\cdots,r_{k+1}\}$ and $I_2=\{r'_0,\cdots,r'_{l+1}\}$ are two critical sequences. If $I_2\subseteq I_1$, $r'_0=r_0$ and $r'_{l+1}=r_{k+1}$, i.e. $I_2=\{r_0,r_{i_1},\cdots,r_{i_l},r_{k+1}\}$, denote them by $I_2\preceq I_1$.

We use the notation Λ_{I_1} to represent the gluing parameter for $\mathcal{M}_{I_1}$. Here $\Lambda_{I_1}=(\lambda_1,\cdots,\lambda_{|I_1|})\in\prod_{i=1}^{|I_1|}[0,+\infty)=[0,+\infty)^{|I_1|}$. By the relation between I_1 and I_2, we introduce the following definitions of the tuples induced from Λ_{I_1}. Define $\Lambda_{I_1,I_2}\in[0,+\infty)^{|I_2|}$ as

$$\Lambda_{I_1,I_2}=(\lambda_{i_1},\cdots,\lambda_{i_l}). \tag{3.1}$$

Here we consider Λ_{I_1,I_2} as a gluing parameter for $\mathcal{M}_{I_2}$. Define $\Lambda_{I_1}(I_1-I_2)\in[0,+\infty)^{|I_1|}$ as

$$\Lambda_{I_1}(I_1-I_2)(i)=\begin{cases}0 & r_i\in I_2,\\ \lambda_i & r_i\notin I_2.\end{cases} \tag{3.2}$$

For example, suppose $I_1=\{r_0,r_1,r_2,r_3,r_4\}$, $I_2=\{r_0,r_2,r_4\}$ and $\Lambda_{I_1}=(5,6,7)$, then $\Lambda_{I_1,I_2}=(6)$ and $\Lambda_{I_1}(I_1-I_2)=(5,0,7)$.

Suppose $I_1=\{r_0,\cdots,r_{k+1}\}$, $I_2=\{r'_0,\cdots,r'_{l+1}\}$ and $r_{k+1}=r'_0$. Define

$$I_1\cdot I_2=\{r_0,\cdots,r_{k+1},r'_1,\cdots,r'_{l+1}\}. \tag{3.3}$$

If $x_1=(a_1,\cdots,a_{k+1})\in\mathcal{M}_{I_1}$ and $x_2=(a'_1,\cdots,a'_{l+1})\in\mathcal{M}_{I_2}$, then define

$$x_1\cdot x_2=(a_1,\cdots,a_{k+1},a'_1,\cdots,a'_{l+1})\in\mathcal{M}_{I_1}\times\mathcal{M}_{I_2}=\mathcal{M}_{I_1\cdot I_2}. \tag{3.4}$$

Suppose $\Lambda_{I_1}=(\lambda_1,\cdots,\lambda_{|I_1|})$ and $\Lambda_{I_2}=(\lambda'_1,\cdots,\lambda'_{|I_2|})$, define

$$\Lambda_{I_1}\cdot\Lambda_{I_2}=(\lambda_1,\cdots,\lambda_{|I_1|},0,\lambda'_1,\cdots,\lambda'_{|I_2|}). \tag{3.5}$$

In particular, if $|I_1|=0$, then $\Lambda_{I_1}\cdot\Lambda_{I_2}=(0,\lambda'_1,\cdots,\lambda'_{|I_2|})$. If $|I_2|=0$, then $\Lambda_{I_1}\cdot\Lambda_{I_2}=(\lambda_1,\cdots,\lambda_{|I_1|},0)$. If $|I_1|=|I_2|=0$, then $\Lambda_{I_1}\cdot\Lambda_{I_2}=(0)$.

Suppose $I=\{r_0,r_1,\cdots,r_{k+1}\}$ is a critical sequence. Recall that an element $x\in\mathcal{M}_I$ is a (un)broken trajectory which is broken at the points r_i $(i=1,\cdots,k)$.

A gluing should be a map $G_I : \mathcal{M}_I \times [0, \epsilon_I)^{|I|} \longrightarrow \overline{\mathcal{M}(r_0, r_{|I|+1})}$ for some $\epsilon_I > 0$. For all $(x, \Lambda_I) \in \mathcal{M}_I \times [0, \epsilon_I)^{|I|}$, we have $\Lambda_I = (\lambda_1, \cdots, \lambda_{|I|})$ is a parameter of gluing, and $G_I(x, \Lambda_I)$ is the (un)broken trajectory glued from x. We expect that $G_I(x, \Lambda_I)$ is not broken at r_i if and only if $\lambda_i > 0$. Thus we can interpret the gluing map as a collaring map, which leads to the following definition.

Definition 3.1. A map $G_I : \mathcal{M}_I \times [0, \epsilon_I)^{|I|} \to \overline{\mathcal{M}(r_0, r_{|I|+1})}$ for some $\epsilon_I > 0$ is a gluing map if it satisfies the following properties. (1). It is a smooth embedding. In particular, if $|I| = 0$, $G_I : \mathcal{M}_I = \mathcal{M}(r_0, r_1) \to \overline{\mathcal{M}(r_0, r_1)}$ is the inclusion. (2). It satisfies the stratum condition, i.e., suppose $I = \{r_0, r_1, \cdots, r_{k+1}\}$, $\Lambda_I = (\lambda_1, \cdots, \lambda_{|I|}) \in [0, \epsilon_I)^{|I|}$, $I_1 \preceq I$, and $\lambda_i = 0$ if and only if $r_i \in I_1$, then for all $x \in \mathcal{M}_I$, we have $G_I(x, \Lambda_I) \in \mathcal{M}_{I_1}$.

Now we give two examples to illustrate the compatibility issue of gluing.

Suppose the gluing maps are defined for all critical sequences. Suppose $I_1 = \{r_0, r_1, r_2, r_3, r_4\}$, $I_2 = \{r_0, r_2, r_4\}$, $\Lambda_{I_1} = (\lambda_1, \lambda_2, \lambda_3)$, $\lambda_1 > 0$, $\lambda_3 > 0$, and $x \in \mathcal{M}_{I_1}$. Gluing x at the points r_1 and r_3 at first, we get $y = G_{I_1}(x, \lambda_1, 0, \lambda_3) \in \mathcal{M}_{I_2}$. Do we have $G_{I_2}(y, \lambda_2) = G_{I_1}(x, \lambda_1, \lambda_2, \lambda_3)$? This is a question about the compatibility for a fixed critical pair (r_0, r_4).

Suppose $I_1 = \{r_0, r_1, r_2\}$, $I_2 = \{r_2, r_3, r_4\}$, $\Lambda_{I_1} = (\lambda_1)$, $\Lambda_{I_2} = (\lambda_2)$, $x_1 \in \mathcal{M}_{I_1}$ and $x_2 \in \mathcal{M}_{I_2}$. Gluing x_1 and x_2, we get $y_1 = G_{I_1}(x_1, \lambda_1) \in \mathcal{M}(r_0, r_2)$ and $y_2 = G_{I_2}(x_2, \lambda_2) \in \mathcal{M}(r_2, r_4)$. Do we have $G_{I_1 \cdot I_2}(x_1 \cdot x_2, \lambda_1, 0, \lambda_2) = (y_1, y_2)$? This is a question about the compatibility for different critical pairs.

The following theorem answers the above two questions.

Theorem 3.2. *Under Assumption 3.1, the gluing maps (see Definition 3.1) can be defined for all critical sequences. They satisfy the following compatibility:*

(1). (Compatibility for one critical Pair). Suppose $I_2 \preceq I_1$, let $\epsilon = \min\{\epsilon_{I_1}, \epsilon_{I_2}\}$. Then, for all $x \in \mathcal{M}_{I_1}$ and $\Lambda_{I_1} = (\lambda_1, \cdots, \lambda_{|I_1|}) \in [0, \epsilon)^{|I_1|}$ such that $\lambda_i > 0$ when $r_i \notin I_2$, we have

$$G_{I_1}(x, \Lambda_{I_1}) = G_{I_2}(G_{I_1}(x, \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1, I_2}). \tag{3.6}$$

(2). (Compatibility for Critical Pairs). Suppose $I_1 = \{r_0, \cdots, r_{k+1}\}$ and $I_2 = \{r_{k+1}, \cdots, r_n\}$. Let $\epsilon = \min\{\epsilon_{I_1}, \epsilon_{I_2}, \epsilon_{I_1 \cdot I_2}\}$, then for all $x_1 \in \mathcal{M}_{I_1}$, $x_2 \in \mathcal{M}_{I_2}$, $\Lambda_{I_1} \in [0, \epsilon)^{|I_1|}$, and $\Lambda_{I_2} \in [0, \epsilon)^{|I_2|}$, we have

$$\begin{aligned} G_{I_1 \cdot I_2}(x_1 \cdot x_2, \Lambda_{I_1} \cdot \Lambda_{I_2}) &= (G_{I_1}(x_1, \Lambda_{I_1}), G_{I_2}(x_2, \Lambda_{I_2})) \\ &\in \overline{\mathcal{M}(r_0, r_{k+1})} \times \overline{\mathcal{M}(r_{k+1}, r_n)}. \end{aligned} \tag{3.7}$$

Theorem 3.2 will follow from a more general Theorem 4.4.

We introduce a traditional notation of gluing as in the Introduction (see e.g. [15, p. 529]). Suppose $\gamma_1 \in \mathcal{M}(p, r)$ and $\gamma_2 \in \mathcal{M}(r, q)$ are two trajectories. We denote the gluing map $G_{\{p,r,q\}}(\gamma_1, \gamma_2, \lambda)$ by $\gamma_1 \#_\lambda \gamma_2$. From Theorem 3.2 we immediately derive the following.

Theorem 3.3. *Under Assumption 3.1, there exist $\epsilon_I > 0$ for all critical sequences I with $|I| = 1$ or $|I| = 2$. For all $\{r_0, r_1, r_2\}$, the gluing $\gamma_1 \#_\lambda \gamma_2$ can be defined for $(\gamma_1, \gamma_2) \in \mathcal{M}(r_0, r_1) \times \mathcal{M}(r_1, r_2)$ and $\lambda \in [0, \epsilon_{\{r_0, r_1, r_2\}})$. The gluing satisfies the following associativity:*

For all $\gamma_1 \in \mathcal{M}(p_1, p_2)$, $\gamma_2 \in \mathcal{M}(p_1, p_2)$, $\gamma_3 \in \mathcal{M}(p_2, p_3)$, and λ_1, $\lambda_2 \in (0, \epsilon)$, where $\epsilon = \min\{\epsilon_{\{p_0, p_1, p_2\}}, \epsilon_{\{p_1, p_2, p_3\}}, \epsilon_{\{p_0, p_1, p_2, p_3\}}\}$, we have

$$(\gamma_1 \#_{\lambda_1} \gamma_2) \#_{\lambda_2} \gamma_3 = \gamma_1 \#_{\lambda_1} (\gamma_2 \#_{\lambda_2} \gamma_3). \tag{3.8}$$

Proof.

$$\begin{aligned} & (\gamma_1 \#_{\lambda_1} \gamma_2) \#_{\lambda_2} \gamma_3 \\ = \; & G_{\{p_0,p_2,p_3\}} \left(G_{\{p_0,p_1,p_2\}}(\gamma_1, \gamma_2, \lambda_1), \gamma_3, \lambda_2 \right) \\ = \; & G_{\{p_0,p_2,p_3\}} \left(G_{\{p_0,p_1,p_2,p_3\}}(\gamma_1, \gamma_2, \gamma_3, \lambda_1, 0), \lambda_2 \right) \\ = \; & G_{\{p_0,p_1,p_2,p_3\}}(\gamma_1, \gamma_2, \gamma_3, \lambda_1, \lambda_2). \end{aligned}$$

Here we have used the (2) of Theorem 3.2 in the second equality and the (1) of Theorem 3.2 in the third equality.

Similarly,

$$\gamma_1 \#_{\lambda_1} (\gamma_2 \#_{\lambda_2} \gamma_3) = G_{\{p_0,p_1,p_2,p_3\}}(\gamma_1, \gamma_2, \gamma_3, \lambda_1, \lambda_2).$$

This completes the proof. □

Remark 3.2. Suppose $I = \{r_0, \cdots, r_{n+1}\}$ is a critical sequence. Let $\epsilon = \min\{\epsilon_J \mid J \subseteq I, \text{ and } |J| = 1 \text{ or } 2.\}$. Then, for $(\gamma_1, \gamma_2) \in \mathcal{M}(r_i, r_j) \times \mathcal{M}(r_j, r_k)$, the gluing $\gamma_1 \#_\lambda \gamma_2$ in Theorem 3.3 can be defined for $\lambda \in [0, \epsilon)$. And the gluing satisfies the associativity. Thus we can define G_J on $\mathcal{M}_J \times (0, \epsilon)^{|J|}$ for any $J \subseteq I$ by inductive gluing of pairs of trajectories. The definition of G_J does not depend on the order of the pairwise gluing.

By [20, Theorem 3.3] and [21, Theorem 7.5], Assumption 3.1 holds in certain cases. Thus Theorems 3.2 and 3.3 lead to the following two propositions. See [20, Definition 2.16] for the definition of a locally trivial metric.

Proposition 3.4. *Suppose M is a complete Hilbert-Riemannian manifold equipped with Morse function f satisfying Condition (C) and having finite indices. Assume that the metric on M is locally trivial and $-\nabla f$ satisfies transversality. Give $\mathcal{M}(p, q)$ the smooth structure induced from $\mathcal{D}(p)$ and $\mathcal{A}(q)$. Then there exist smooth structures on $\overline{\mathcal{M}(p, q)}$ and gluing maps which satisfy the compatibility and associativity in Theorems 3.2 and 3.3.*

Proposition 3.5. *Suppose M is a compact Riemannian manifold equipped with with Morse function f. Assume $-\nabla f$ satisfies tranversality. Then there exist smooth structures on $\mathcal{M}(p, q)$ and $\overline{\mathcal{M}(p, q)}$ and gluing maps which satisfy the compatibility and associativity in Theorems 3.2 and 3.3.*

Remark 3.3. Proposition 3.5 is based on [21]. In the case of a compact M, it has the advantage that the metric is allowed to be general. However, the smooth structure on $\mathcal{M}(p, q)$ may be different from the natural one when the metric is not locally trivial.

4. Generalization

The proof of Theorem 3.2 actually does not directly depend on the speciality of Morse theory. Therefore, we will generalize the results to Theorem 4.4 which is about collaring maps of manifolds with faces.

Definition 4.1. An n-dimensional *smooth manifold with corners* is a space defined in the same way as a smooth manifold except that its atlases are open subsets of $[0, +\infty)^n$.

If L is a smooth manifold with corners, $x \in L$, a neighborhood of x is diffeomorphic to $(0,\epsilon)^{n-k} \times [0,\epsilon)^k$, then define $c(x) = k$. Clearly, $c(x)$ does not depend on the choice of atlas.

Definition 4.2. Suppose L is a smooth manifold. We call $\{x \in L \mid c(x) = k\}$ the k-stratum of L. Denote it by $\partial^k L$.

Clearly, $\partial^k L$ is a submanifold *without* corners inside L, its codimension is k.

Definition 4.3. (c.f. [17]). A *smooth manifold L with faces* is a smooth manifold with corners such that each x belongs to the closures of $c(x)$ different components of $\partial^1 L$.

Now we introduce the notation Ω, " $\succeq$ ", I and $\mathcal{M}(p,q)$ as in Section 3. However, in the present context they are generalizations: they are *independent* of Morse theory.

Suppose Ω is a partially ordered set with a partial order " $\succeq$ ". Suppose $I = \{r_0, r_1, \cdots, r_{k+1}\}$ is a finite chain of Ω, i.e., $I \subseteq \Omega$ and $r_i \succ r_{i+1}$. We call r_0 the head of I and r_{k+1} the tail of I. Define the length of I as $|I| = k$. If $J \subseteq I$, $J = \{r'_0, \cdots, r'_{l+1}\}$, $r'_0 = r_0$ and $r'_{l+1} = r_{k+1}$, i.e. $J = \{r_0, r_{i_1}, \cdots, r_{i_l}, r_{k+1}\}$, denote them by $J \preceq I$. Suppose $I_1 = \{r_0, \cdots, r_{k+1}\}$ and $I_2 = \{r_{k+1}, \cdots, r_n\}$ are two chains. Define $I_1 \cdot I_2 = \{r_0, \cdots, r_n\}$, which is also a chain.

Suppose a finite dimensional manifold $\mathcal{M}(p,q)$ is defined for each pair $(p,q) \subseteq \Omega$ such that $p \succ q$. For the above chain I, define $\mathcal{M}_I = \prod_{i=0}^{|I|} \mathcal{M}(r_i, r_{i+1})$.

Assumption 4.1. *The partially ordered set Ω is countable. The finite dimensional manifolds $\mathcal{M}(p,q)$ can be compactified to be $\overline{\mathcal{M}(p,q)}$ which are compact smooth manifolds with faces. These $\overline{\mathcal{M}(p,q)}$ satisfy the following conditions:*

(1). We have $\overline{\mathcal{M}(p,q)} = \bigsqcup_I \mathcal{M}_I$, where the disjoint is over all finite chains I with head p and tail q. The k-stratum of $\overline{\mathcal{M}(p,q)}$ is $\bigsqcup_{|I|=k} \mathcal{M}_I$, and each $\mathcal{M}_I$ is an open subset of the k-stratum. The smooth structure of $\overline{\mathcal{M}(p,q)}$ is compatible with those of $\mathcal{M}_I$.

(3). Suppose $p \succ r \succ q$, then the natural inclusion $\overline{\mathcal{M}(p,r)} \times \overline{\mathcal{M}(r,q)} \hookrightarrow \overline{\mathcal{M}(p,q)}$ is a smooth embedding.

We introduce the following definitions similar to Section 3. Use $\Lambda_I = (\lambda_1, \cdots, \lambda_{|I|})$ to represent the collaring parameter for $\mathcal{M}_I$. Define $\Lambda_{I_1}(I_1 - I_2)$, Λ_{I_1,I_2} and $\Lambda_{I_1} \cdot \Lambda_{I_2}$. Also for $x_1 \in \mathcal{M}_{I_1}$ and $x_2 \in \mathcal{M}_{I_2}$, define $x_1 \cdot x_2 \in \mathcal{M}_{I_1 \cdot I_2}$.

Define the collaring map $G_I : \mathcal{M}_I \times [0, \epsilon_I)^{|I|} \to \overline{\mathcal{M}(r_0, r_{|I|+1})}$ as Definition 3.1.

The proof of the following theorem is given in Section 6.

Theorem 4.4. *Under Assumption 4.1, the collaring maps G_I can be defined for all finite chains I of Ω. These maps satisfy the following compatibility:*

(1). Suppose $I_2 \preceq I_1$, let $\epsilon = \min\{\epsilon_{I_1}, \epsilon_{I_2}\}$. Then, for all $x \in \mathcal{M}_{I_1}$ and $\Lambda_{I_1} \in [0,\epsilon)^{|I_1|}$ such that $\lambda_i > 0$ when $r_i \notin I_2$, we have

$$G_{I_1}(x, \Lambda_{I_1}) = G_{I_2}(G_{I_1}(x, \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1,I_2}). \tag{4.1}$$

(2). Suppose $I_1 = \{r_0, \cdots, r_{k+1}\}$ and $I_2 = \{r_{k+1}, \cdots, r_n\}$. Let $\epsilon = \min\{\epsilon_{I_1}, \epsilon_{I_2}, \epsilon_{I_1 \cdot I_2}\}$, then for all $x_1 \in \mathcal{M}_{I_1}$, $x_2 \in \mathcal{M}_{I_2}$, $\Lambda_{I_1} \in [0,\epsilon)^{|I_1|}$, and $\Lambda_{I_2} \in [0,\epsilon)^{|I_2|}$, we have

$$G_{I_1 \cdot I_2}(x_1 \cdot x_2, \Lambda_{I_1} \cdot \Lambda_{I_2}) = (G_{I_1}(x_1, \Lambda_{I_1}), G_{I_2}(x_2, \Lambda_{I_2})). \tag{4.2}$$

5. Face Structures

In order to prove Theorem 4.4, we first study the face structures.

Suppose L is manifold with faces. The closure of a component of $\partial^1 L$ (see Definition 4.2) is still connected. Following the terminology of [17], we have the following definition.

Definition 5.1. We call the closure of a component of $\partial^1 L$ a connected (closed) face of L. We call any union of pairwise disjoint connected faces a face of L.

Thus, if F is a face of L, then $F = \bigsqcup_{\alpha \in \mathfrak{A}} C_\alpha$, where C_α is the closure of C_α° and C_α° is a component of $\partial^1 L$. As pointed in [17], F is still a manifold with corners. We have the following result which is trivial when $\mathfrak{A}$ is a finite set.

Lemma 5.2. *Using the notation as the above, we have that F is a smoothly embedded submanifold with corners inside L. The components of F are C_α. The interior of F (i.e. $\partial^0 F$) is $\bigsqcup_{\alpha \in \mathfrak{A}} C_\alpha^\circ$ and F is a closed subset of L.*

Proof. First, we show that C_α is a submanifold with corners and its 0-stratum is C_α°. It suffices to show that, for each $x \in C_\alpha$, there exists an open neighborhood U_x of x such that $U_x \cap C_\alpha$ has the desired corner structure.

We can choose U_x such that it has the chart $(-\epsilon, \epsilon)^{n-l} \times [0, \epsilon)^l$ and x has the coordinate $(0, \cdots, 0)$. Clearly,

$$U_x \cap C_\alpha^\circ \subseteq \bigsqcup_{i=1}^{l} \left[(-\epsilon, \epsilon)^{n-l} \times (0, \epsilon)^{i-1} \times \{0\} \times (0, \epsilon)^{l-i}\right],$$

and $U_x \cap C_\alpha^\circ \neq \emptyset$. We may assume $[(-\epsilon, \epsilon)^{n-l} \times \{0\} \times (0, \epsilon)^{l-1}] \cap C_\alpha^\circ \neq \emptyset$. Since $(-\epsilon, \epsilon)^{n-l} \times \{0\} \times (0, \epsilon)^{l-1}$ is connected and contained in $\partial^1 L$, and C_α° is a component of $\partial^1 L$, we infer that $(-\epsilon, \epsilon)^{n-l} \times \{0\} \times (0, \epsilon)^{l-1} \subseteq C_\alpha^\circ$. By Definition 4.3, it's easy to see $U_x \cap C_\alpha^\circ = (-\epsilon, \epsilon)^{n-l} \times \{0\} \times (0, \epsilon)^{l-1}$. Since, U_x is open, we have $U_x \cap C_\alpha$ is the relative closure of $U_x \cap C_\alpha^\circ$ in U_x. In other words, $U_x \cap C_\alpha = (-\epsilon, \epsilon)^{n-l} \times \{0\} \times [0, \epsilon)^{l-1}$ and the 0-stratum of $U_x \cap C_\alpha$ is contained in C_α°. Thus we get the desired corner structure.

Second, we show that F is a manifold with corners.

Since C_α has no intersection with other C_β, by the above argument, we can see that the above open neighborhood U_x has no intersection with other C_β. Thus $\bigcup_{x \in C_\alpha} U_x$ is an open neighborhood of C_α which has no intersection with other C_β. So C_α is relatively open in F. This verifies the manifold structure of F.

Finally, we show that F is a closed subset of L. Suppose x is in the closure of F, then x can be approximated by points in F and thus by points in $\bigsqcup_{\alpha \in \mathfrak{A}} C_\alpha^\circ$. By the above argument, it's easy to see that x belongs to some C_α. $\square$

Lemma 5.3. *Suppose L is an n dimensional manifold with faces. Suppose F_i ($i = 1, \cdots k$) are faces of L such that their interiors are pairwise disjoint and $\bigcap_{i=1}^k F_i$ is nonempty. Then $\bigcap_{i=1}^k F_i$ is an $n-k$ dimensional smoothly embedded submanifold with corners inside L.*

Proof. Let x be an arbitrary point in $\bigcap_{i=1}^k F_i$. It suffices to prove that there exists an open neighborhood U of x such that $U \cap \bigcap_{i=1}^k F_i$ has a corner structure.

For each i, x belongs to an unique component of F_i. Since this component is relatively open in F_i, we can choose U small enough such that U has no intersection with other components. Thus we may assume F_i is connected.

By the proof of Lemma 5.2, we can choose U such that it has a chart $(-\epsilon,\epsilon)^{n-l}\times[0,\epsilon)^l$, x has the coordinate $(0,\cdots,0)$ and $U\cap F_1=(-\epsilon,\epsilon)^{n-l}\times\{0\}\times[0,\epsilon)^{l-1}$. Since the interior of F_i are pairwise disjoint, repeating this argument, we get $U\cap F_i=(-\epsilon,\epsilon)^{n-l}\times[0,\epsilon)^{i-1}\times\{0\}\times[0,\epsilon)^{l-i}$. Thus $U\cap\bigcap_{i=1}^k F_i=(-\epsilon,\epsilon)^{n-l}\times\{0\}^k\times[0,\epsilon)^{l-k}$. This verifies the corner structure. □

We introduce some other concepts following [14].

Definition 5.4. Suppose L is a manifold with corners. For all $x\in L$,

$$A_xL=\{v\in T_xL\mid v=\gamma'(0)\text{ for some smooth curve }\gamma:[0,\epsilon)\longrightarrow L.\}$$

is the tangent sector of L at x.

Definition 5.4 is equivalent to the *secteur tangent* in [14, p. 3].

Definition 5.5. Suppose L_1 is a submanifold *without* corners inside L and $x\in L_1$, we define the normal sector $A_x(L_1,L)=A_xL/T_xL_1$.

In [14], $A_x(L_1,L)$ is called *secteur transverse*.

Define the tangent sector bundle AL as the subbundle of TL with fibers A_xL. Define the normal bundle $N(L_1,L)$ as the bundle whose fibers are the normal space $N_x(L_1,L)=T_xL/T_xL_1$. Define the normal sector bundle $A(L_1,L)$ as the subbundle of $N(L_1,L)$ with fiber $A_x(L_1,L)$ and $A_{L_1}L$ as the restriction of AL to L_1.

Lemma 5.6. *Under the assumption of Lemma 5.3, assume that L_1 is an open subset of ∂^kL and $L_1\subseteq\bigcap_{i=1}^kF_i$. Then there exist smooth sections e_i of $A_{L_1}L$ ($i=1,\cdots,k$) satisfying the following stratum condition: (1). $e_i\in A_{L_1}(\bigcap_{j\neq i}F_j)$; (2). $\{\pi e_1,\cdots,\pi e_k\}$ is linearly independent everywhere and all elements in $A_x(L_1,L)$ can be linearly represented by $\{\pi e_1(x),\cdots,\pi e_k(x)\}$ with nonnegative coefficients, where $\pi:A_{L_1}L\to A(L_1,L)$ is the natural projection.*

Proof. Suppose $x\in L_1$, by the proof of Lemma 5.3, there exists a neighborhood U of x such that U has a chart $(-\epsilon,\epsilon)^{n-k}\times[0,\epsilon)^k$, x has the coordinate $(0,\cdots,0)$, $U\cap L_1=(-\epsilon,\epsilon)^{n-k}\times\{0\}^k$ and $U\cap F_i=(-\epsilon,\epsilon)^{n-k}\times[0,\epsilon)^{i-1}\times\{0\}\times[0,\epsilon)^{k-i}$. Thus $U\cap\bigcap_{j\neq i}F_j=(-\epsilon,\epsilon)^{n-k}\times\{0\}^{i-1}\times[0,\epsilon)\times\{0\}^{k-i}$. Obviously, for any vector $e_i(x)\in A_x(\bigcap_{j\neq i}F_j)-T_xL_1$, we have $\{\pi e_1(x),\cdots,\pi e_k(x)\}$ satisfies the desired property in $A_x(L_1,L)$.

Since L_1 is an open subset of the 1-stratum of $\bigcap_{j\neq i}F_j$, we can choose a smooth inward normal section e_i along L_1. □

In the case of Assumption 4.1, it's easy to see that $\overline{\mathcal{M}(p,q)}$ is a manifold with faces $\overline{\mathcal{M}(p,r)}\times\overline{\mathcal{M}(r,q)}$. The interiors of these faces are $\mathcal{M}(p,r)\times\mathcal{M}(r,q)$ which are pairwise disjoint. Suppose $I=\{p,r_1,\cdots,r_k,q\}$ is a chain of Ω. Let $I_i=\{p,r_1,\cdots,r_{i-1},r_{i+1},\cdots,r_k,q\}$. Then $\mathcal{M}_I$ is the interior of $\bigcap_{i=1}^k\overline{\mathcal{M}(p,r_i)}\times\overline{\mathcal{M}(r_i,q)}$, and $\bigcap_{j\neq i}\overline{\mathcal{M}(p,r_j)}\times\overline{\mathcal{M}(r_j,q)}=\overline{\mathcal{M}_{I_i}}$. By Lemma 5.6, we have the following corollary.

Corollary 5.7. *There exists a smooth frame $\{e_1,\cdots,e_k\}$ along $\mathcal{M}_I$ satisfying the following stratum condition: (1). $e_i\in A_{\mathcal{M}_I}\overline{\mathcal{M}_{I_i}}$; (2). $\{\pi e_1,\cdots,\pi e_k\}$ is linearly independent everywhere and all elements in $A_x(\mathcal{M}_I,\overline{\mathcal{M}(p,q)})$ can be linearly represented by $\{\pi e_1(x),\cdots,\pi e_k(x)\}$ with nonnegative coefficients, where $\pi:A_{\mathcal{M}_I}\overline{\mathcal{M}(p,q)}\to A(\mathcal{M}_I,\overline{\mathcal{M}(p,q)})$ is the natural projection.*

For a manifold L with corners, [14, p. 8] shows that there exists a connection on L such that all strata are totally geodesic. (See [4, Chapter 4] for a detailed treatment of connections.) Suppose L_1 is a stratum of L. Then by the above connection and the exponential map, [14] shows that an open neighborhood of L_1 in $A(L_1, L)$ is diffeomorphic to an open neighborhood of L_1 in L. Thus by the frame in Corollary 5.7, we get the following lemma.

Lemma 5.8. *There is a smooth embedding $\varphi_I : \mathcal{M}_I \times [0,1)^{|I|} \longrightarrow \overline{\mathcal{M}(p,q)}$ satisfying the stratum condition (See (2) in Definition 3.1).*

In order to prove Theorem 4.4, we need some connections even better than the above one. This leads to the definition of the product connection. There are several ways to define a connection on a manifold L. One is as follows. A connection is to assign each smooth curve $\gamma : [0,1] \longrightarrow L$ a parallel transport (or displacement) $P_\gamma : T_{\gamma(0)}L \longrightarrow T_{\gamma(1)}L$ which is a linear isomorphism. Suppose L_1 and L_2 are two manifolds with corners. Clearly, $T(L_1 \times L_2) = TL_1 \times TL_2$. We define the product connection on $L_1 \times L_2$ as follows.

Definition 5.9. Let $\gamma = (\gamma_1, \gamma_2) : [0,1] \longrightarrow L_1 \times L_2$ be a smooth curve. Define the parallel transport $P_\gamma : T_{\gamma(0)}(L_1 \times L_2) \longrightarrow T_{\gamma(1)}(L_1 \times L_2)$ as $P_\gamma(v_1, v_2) = (P_{\gamma_1} v_1, P_{\gamma_2} v_2)$, where P_{γ_i} is the parallel transport along γ_i. The connection assigning P_γ is the product connection.

For a product connection, a curve γ in $L_1 \times L_2$ is a geodesic if and only if both γ_1 and γ_2 are geodesics. By Lemma 5.8, φ_I pulls back the connection on $\overline{\mathcal{M}(p,q)}$ to $\mathcal{M}_I \times [0,1)^{|I|}$. Let γ be a curve in $\mathcal{M}_I \times [0,1)^{|I|}$ such that $\gamma(t) = (x, \sigma(t))$, where $x \in \mathcal{M}_I$ and σ is a straight line in $[0,1)^{|I|}$. If σ passes through the origin, then γ is a geodesic because φ_I is defined by the exponential map. Since $\mathcal{M}_I$ is totally geodesic in $\overline{\mathcal{M}(p,q)}$, we infer that $\mathcal{M}_I$ has a connection. Moreover, $[0,1)^{|I|}$ also has its standard flat connection. We can define the product connection of $\mathcal{M}_I \times [0,1)^{|I|}$. The product connection coincides with the old one on $T(\mathcal{M}_I \times \{0\}^{|I|})$, and φ_I is still given by the exponential map under the new connection. This new connection has its advantage over the old one. In particular, for *every* straight line σ in $[0,1)^{|I|}$, *not* necessarily passing through the origin, $\gamma(t) = (x, \sigma(t))$ is a geodesic of the new connection. This is important in the proof of Theorem 4.4.

6. Proof of Theorem 4.4

Before proving Theorem 4.4, we shall introduce some definitions and notation.

Definition 6.1. Suppose $(p,q) \subseteq \Omega$, where Ω is the set defined in Assumption 4.1. If $p \not\succ q$, then define the length of (p,q) as $|p,q| = -1$. Otherwise, define the length of (p,q) as $|p,q| = \sup\{|I| \mid I \text{ is a chain with head } p \text{ and tail } q\}$.

By (1) of Assumption 4.1, we know that $|p,q| \leq \dim(\overline{\mathcal{M}(p,q)}) < +\infty$.

By the compactness of $\overline{\mathcal{M}(p,q)}$ and (1) of Assumption 4.1, there are only finitely many chains I with head p and tail q.

Suppose $I_1 = \{r_0, \cdots, r_{k+1}\}$ and $I_2 = \{r_0, r_{i_1}, \cdots, r_{i_l}, r_{k+1}\}$ are two chains of Ω such that $I_2 \preceq I_1$. Like Section 3, if $\Lambda_{I_2} = (\lambda_{i_1}, \cdots, \lambda_{i_l}) \in [0, +\infty)^{|I_2|}$ is a collaring parameter for for $\mathcal{M}_{I_2}$, then define $\Lambda_{I_2, I_1} \in [0, +\infty)^{|I_1|}$ as

$$\Lambda_{I_2,I_1}(i) = \begin{cases} \lambda_i & r_i \in I_2, \\ 0 & r_i \notin I_2. \end{cases} \tag{6.1}$$

Here we consider Λ_{I_2,I_1} as a collaring parameter for $\mathcal{M}_{I_1}$.

If $I_i \prec I$ $(i = 1, \cdots, n)$, then define

$$\Lambda_I + \Lambda_{I_1} + \cdots \Lambda_{I_n} = \Lambda_I + \Lambda_{I_1,I} + \cdots \Lambda_{I_n,I}. \tag{6.2}$$

Clearly,

$$\Lambda_{I_1} = \Lambda_{I_1}(I_1 - I_2) + \Lambda_{I_1,I_2}$$

For example, suppose $I_1 = \{r_0, r_1, r_2, r_3, r_4\}$, $I_2 = \{r_0, r_2, r_4\}$ and $\Lambda_{I_1} = (5, 6, 7)$, then $\Lambda_{I_1,I_2} = (6)$, and

$$\Lambda_{I_1}(I_1 - I_2) + \Lambda_{I_1,I_2} = (5, 0, 7) + (6) = (5, 0, 7) + (0, 6, 0) = (5, 6, 7) = \Lambda_{I_1}.$$

If $\Lambda_{I_2} = (8)$, then $\Lambda_{I_2,I_1} = (0, 8, 0)$ and

$$\Lambda_{I_1} + \Lambda_{I_2} = (5, 6, 7) + (8) = (5, 6, 7) + (0, 8, 0) = (5, 14, 7).$$

Proof of Theorem 4.4. We shall define G_I by exponential maps. This requires two things. First, we need a frame satisfying the stratum condition (See Corollary 5.7) in $A(\mathcal{M}_I, \overline{\mathcal{M}(p,q)})$. Second, we need a connection on $\overline{\mathcal{M}(p,q)}$. The proof is to construct the above two things by a double induction. The outer induction is on the length $|p,q|$. We construct the desired G_I in the case of $|p,q| = n$ based on the hypothesis that all G_I have been constructed and satisfy (4.1) and (4.2) for all $|p,q| < n$. The inner induction is the process to construct G_I for a fixed pair (p,q).

(1). The first step of the outer induction (the induction on $|p,q|$).

When $|p,q| = 0$, then $\mathcal{M}_I = \overline{\mathcal{M}(p,q)}$, define $G_I : \mathcal{M}_I \to \overline{\mathcal{M}(p,q)}$ as the identity.

(2). The second step of the outer induction (the induction on $|p,q|$).

Suppose we have constructed the desired G_I for all pair (p,q) such that $|p,q| < n$. We shall construct G_I in the case of $|p,q| = n$. The construction is the inner induction. Let X_k be the union of all l-strata of $\overline{\mathcal{M}(p,q)}$ with $l \geq k$. Clearly, $X_{k+1} \subseteq X_k$, X_1 is the full boundary of $\overline{\mathcal{M}(p,q)}$. We shall construct a family of open sets U_k such that $U_{k+1} \subseteq U_k$ and $X_k \subseteq U_k$ by an downward induction on k. In other words, we construct U_k after having constructed U_{k+1}. For each k, we shall construct $G_I : (\mathcal{M}_I \cap U_k) \times [0,\epsilon)^{|I|} \to \overline{\mathcal{M}(p,q)}$ such that $\mathrm{Im} G_I \subseteq U_k$, and all G_I satisfy (4.1) and (4.2). We call such a map G_I in U_k, denote it by $G_I|_{U_k}$. Extend G_I with the step of the inner induction. Clearly, U_1 contains all $\mathcal{M}_I$ such that $|I| > 0$. If the construction of $G_I|_{U_1}$ is finished, we shall complete the proof by defining $G_{\{p,q\}}$ as the inclusion.

Since $|p,q| = n$, the stratum with the lowest dimension is the n-stratum.

(I). The first step of the inner induction (the induction on U_k).

We shall construct U_n, $G_I|_{U_n}$, frames for $\mathcal{M}_I \cap U_n$ and a connection providing all G_I via the exponential map. Moreover, $(\mathcal{M}_I \cap U_n) \times [0,\epsilon)^{|I|}$ will also have a product connection (see Definition 5.9 and the comment following it) if we pull back the connection on U_n via G_I.

We know that $X_n = \bigcup_{|J|=n} \mathcal{M}_J$. By Lemma 5.8, we can construct a smooth embedding $\varphi_J : \mathcal{M}_J \times [0,\epsilon_0)^{|J|} \to \overline{\mathcal{M}(p,q)}$ satisfying the stratum condition (See (2) in Definition 3.1). Furthermore, $\mathcal{M}_J$ is compact because it is closed (also open) in the lowest dimensional stratum. Choose ϵ_0 small enough so that $\mathrm{Im}\varphi_J$ are pairwise disjoint for all J such that $|J| = n$. Fix $J = \{p, r_1, \cdots, r_n, q\}$. Suppose $J_l = \{p, r_1, \cdots, r_l\}$ and $J'_l = \{r_l, \cdots, r_n, q\}$. Clearly, $|p, r_l| < n$ and $|r_l, q| < n$. By the outer induction on $|p,q|$, G_{J_l} and $G_{J'_l}$ have been defined.

Lemma 6.2. *There exists $\epsilon > 0$. And φ_J can be modified to be defined on $\mathcal{M}_J \times [0,\epsilon)^{|J|}$ such that for all $l \in \{1, \cdots, n\}$, we have*

$$\varphi_J(x_1 \cdot x_2, \Lambda_{J_l} \cdot \Lambda_{J'_l}) = (G_{J_l}(x_1, \Lambda_{J_l}), G_{J'_l}(x_2, \Lambda_{J'_l})).$$

Proof. For small ϵ, $G_{J_l} \times G_{J'_l}(\mathcal{M}_J \times \prod_{i=1, i\neq l}^{|J|}[0,\epsilon)) \subseteq \mathrm{Im}\varphi_J$, where $G_{J_l} \times G_{J'_l}(x_1 \cdot x_2, \Lambda_{J_l}, \Lambda_{J'_l}) = (G_{J_l}(x_1, \Lambda_{J_l}), G_{J'_l}(x_2, \Lambda_{J'_l}))$.

Consider the following map $\phi_l = \varphi_J^{-1} \circ (G_{J_l} \times G_{J'_l})$,

$$\phi_l : \mathcal{M}_J \times \prod_{i=1, i\neq l}^{|J|} [0,\epsilon) \to \mathrm{Im}\varphi_J \to \mathcal{M}_J \times [0,\epsilon_0)^{|J|}.$$

We only need to prove that φ_J can be modified such that for all l,

$$\phi_l(x, \lambda_1, \cdots, \lambda_{l-1}, \lambda_{l+1}, \cdots, \lambda_n) = (x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_n). \tag{6.3}$$

Denote $(\lambda_1, \cdots, \lambda_n)$ by Λ_J, $(\lambda_1, \cdots, \lambda_{l-1}, \lambda_{l+1}, \cdots, \lambda_n)$ by Λ_{J-l}, $(\lambda_1, \cdots, \lambda_{l-1})$ by Λ_{J_l}, and $(\lambda_{l+1}, \cdots, \lambda_n)$ by $\Lambda_{J'_l}$. Since $\mathrm{Im}(G_{J_l} \times G_{J'_l}) \subseteq \overline{\mathcal{M}(p, r_l)} \times \overline{\mathcal{M}(r_l, q)}$ and φ_J satisfies the stratum condition, we have

$$\phi_l(x, \Lambda_{J-l}) = (a, c_1, \cdots, c_{l-1}, 0, c_{l+1}, \cdots, c_n)$$

where a and c_i are smooth functions of x and Λ_{J-l}.

Define $\theta_l : \mathcal{M}_J \times [0,\epsilon)^{|J|} \to \mathcal{M}_J \times [0,\epsilon_0)^{|J|}$ as

$$\theta_l(x, \Lambda_J) = (a, \cdots, c_{l-1}, \lambda_l, c_{l+1}, \cdots, c_n). \tag{6.4}$$

Since ϕ_l is a smooth embedding, so is θ_l. Since $\mathcal{M}_J$ is compact, shrink ϵ_0 if necessary, we may assume θ_l^{-1} can be defined on $\mathcal{M}_J \times [0,\epsilon_0)^{|J|}$. Thus

$$\begin{aligned} & (\varphi_J \circ \theta_l)^{-1} \circ (G_{J_l} \times G_{J'_l})(x, \Lambda_{J-l}) \\ = \; & \theta_l^{-1} \circ \phi_l(x, \Lambda_{J-l}) \\ = \; & (x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_n) \\ = \; & (x, \Lambda_{J_l} \cdot \Lambda_{J'_l}). \end{aligned}$$

Modify φ_J to be $\varphi_J \circ \theta_l$, we get (6.3) is true for a fixed l and some $\epsilon > 0$.

In general, suppose we have proved (6.3) is true for $l \in \{1, \cdots, j-1\}$, we shall modify φ_J such that (6.3) is true for all $l \in \{1, \cdots, j\}$. Let $x = x_1 \cdot x_2 \cdot x_3$, where $x_1 = (a_0, \cdots, a_{l-1})$, $x_2 = (a_l, \cdots, a_{j-1})$ and $x_3 = (a_j, \cdots, a_n)$. Denote $\{r_l, \cdots, r_j\}$ by $J_{(l,j)}$ and $(\lambda_{l+1}, \cdots, \lambda_{j-1})$ by $\Lambda_{J_{(l,j)}}$.

$$\phi_j(x, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}}, \Lambda_{J'_j}) = \varphi_J^{-1}(G_{J_j}(x_1 \cdot x_2, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}}), G_{J'_j}(x_3, \Lambda_{J'_j})).$$

Since $|p, r_l| < n$, by the outer inductive hypothesis, G_{J_j} satisfies (4.2). Shrink ϵ if necessary, we have

$$G_{J_j}(x_1 \cdot x_2, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}}) = (G_{J_l}(x_1, \Lambda_{J_l}), G_{J_{(l,j)}}(x_2, \Lambda_{J_{(l,j)}})),$$

Similarly,

$$G_{J'_l}(x_2 \cdot x_3, \Lambda_{J_{(l,j)}} \cdot \Lambda_{J'_j}) = (G_{J_{(l,j)}}(x_2, \Lambda_{J_{(l,j)}}), G_{J'_j}(x_3, \Lambda_{J'_j})).$$

Thus

$$G_{J_j} \times G_{J'_j}(x, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}}, \Lambda_{J'_j}) = G_{J_l} \times G_{J'_l}(x, \Lambda_{J_l}, \Lambda_{J_{(l,j)}} \cdot \Lambda_{J'_j}).$$

Then

$$\begin{aligned} & \phi_j(x, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}}, \Lambda_{J'_j}) \\ = \quad & \varphi_J^{-1} \circ (G_{J_j} \times G_{J'_j})(x, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}}, \Lambda_{J'_j}) \\ = \quad & \varphi_J^{-1} \circ (G_{J_l} \times G_{J'_l})(x, \Lambda_{J_l}, \Lambda_{J_{(l,j)}} \cdot \Lambda_{J'_j}) \\ = \quad & \phi_l(x, \Lambda_{J_l}, \Lambda_{J_{(l,j)}} \cdot \Lambda_{J'_j}). \end{aligned}$$

Since ϕ_l satisfies (6.3), we have $\phi_l(x, \Lambda_{J_l}, \Lambda_{J_{(l,j)}} \cdot \Lambda_{J'_j}) = (x, \Lambda_{J_l} \cdot \Lambda_{J_{(l,j)}} \cdot \Lambda_{J'_j})$, or

$$\begin{aligned} & \phi_j(x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_n) \\ = \quad & (x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_{j-1}, 0, \lambda_{j+1}, \cdots, \lambda_n). \end{aligned}$$

Define $\theta_j : \mathcal{M}_J \times [0, \epsilon)^{|J|} \to \mathcal{M}_J \times [0, \epsilon_0)^{|J|}$ as (6.4), we have

$$\theta_j(x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_n) = (x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_n).$$

The operation of θ_j on $\mathcal{M}_J \times \prod_{i=1, i\neq l}^{|J|} [0, \epsilon) \times \{0\}$ is the identity. Thus

$$\begin{aligned} & (\varphi_J \circ \theta_j)^{-1} \circ (G_{J_l} \times G_{J'_l}) \\ = \quad & \theta_j^{-1} \circ (\varphi_J^{-1} \circ (G_{J_l} \times G_{J'_l})) \\ = \quad & \varphi_J^{-1} \circ (G_{J_l} \times G_{J'_l}) \\ = \quad & \phi_l. \end{aligned}$$

So if we modify φ_J to be $\varphi_J \circ \theta_j$, then ϕ_l $(l < j)$ will not change and still satisfy (6.3). However, ϕ_j may change and must satisfy (6.3) now. Thus we get a new φ_J such that (6.3) is true for $l \in \{1, \cdots, j\}$.

By repeating this process, we finish the proof of this lemma. □

Now we define G_I in $\mathrm{Im}\varphi_J$. If $I \not\preceq J$, then $\mathrm{Im}\varphi_J \cap \mathcal{M}_I = \emptyset$, we don't need to consider it. We assume $I \preceq J$.

For all $y \in \mathrm{Im}\varphi_J \cap \mathcal{M}_I$, there exist $x \in \mathcal{M}_J$ and $\Lambda_J \in [0, \epsilon)^n$ such that $y = \varphi_J(x, \Lambda_J)$ where x and Λ_J are unique and $\lambda_i = 0$ if and only if $r_i \in I$. Define $G_I(y, \Lambda_I) = \varphi_J(x, \Lambda_J + \Lambda_I)$. Since φ_J is a smooth embedding, so is G_I. (Actually, if we identify $Im\varphi_J$ with $\mathcal{M}_J \times [0, \epsilon)^{|J|}$ via φ_J, then G_I has the form $G_I((x, \Lambda_J), \Lambda_I) = (x, \Lambda_J + \Lambda_I)$.)

Lemma 6.3. *The maps G_I satisfy (4.1) in $Im\varphi_J$.*

Proof. Suppose $I_2 \preceq I_1 \preceq J$ and $y \in \mathrm{Im}\varphi_J \cap \mathcal{M}_{I_1}$, we need to show that $G_{I_1}(y, \Lambda_{I_1}) = G_{I_2}(G_{I_1}(y, \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1, I_2})$.

Suppose $y = \varphi_J(x, \Lambda_J)$, we have $G_{I_1}(y, \Lambda_{I_1}) = \varphi_J(x, \Lambda_J + \Lambda_{I_1})$, $G_{I_1}(y, \Lambda_{I_1}(I_1 - I_2)) = \varphi_J(x, \Lambda_J + \Lambda_{I_1}(I_1 - I_2))$, and

$$\begin{aligned} & G_{I_2}(G_{I_1}(y, \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1, I_2}) \\ = \quad & G_{I_2}(\varphi_J(x, \Lambda_J + \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1, I_2}) \\ = \quad & \varphi_J(x, \Lambda_J + \Lambda_{I_1}(I_1 - I_2) + \Lambda_{I_1, I_2}) \\ = \quad & \varphi_J(x, \Lambda_J + \Lambda_{I_1}) = G_{I_1}(y, \Lambda_{I_1}). \end{aligned}$$

This completes the proof of the lemma. □

Lemma 6.4. *The maps G_I satisfy (4.2) in $Im\varphi_J$.*

Proof. Suppose $I \preceq J$, $I = I_1 \cdot I_2$, $y_1 \in \mathcal{M}_{I_1}$, $y_2 \in \mathcal{M}_{I_2}$, and $y_1 \cdot y_2 \in \mathrm{Im}\varphi_J$. We need to show that $G_I(y_1 \cdot y_2, \Lambda_{I_1} \cdot \Lambda_{I_2}) = (G_{I_1}(y_1, \Lambda_{I_1}), G_{I_2}(y_2, \Lambda_{I_2}))$.

Since $I \preceq J$, we have $J = J_l \cdot J_l'$, $I_1 \preceq J_l$ and $I_2 \preceq J_l'$ for some $J_l = \{p, r_1, \cdots, r_l\}$ and $J_l' = \{r_l, \cdots, r_n, q\}$. Since $y_1 \cdot y_2 \in \mathcal{M}_{I_1} \times \mathcal{M}_{I_2}$ and $y_1 \cdot y_2 = \varphi_J(x, \Lambda_J)$, we have $x = x_1 \cdot x_2$ for some $x_1 \in \mathcal{M}_{J_l}$ and $x_2 \in \mathcal{M}_{J_l'}$ and $\Lambda_J = \Lambda_{J_l} \cdot \Lambda_{J_l'}$ for some Λ_{J_l} and $\Lambda_{J_l'}$. Thus $y_1 \cdot y_2 = \varphi_J(x_1 \cdot x_2, \Lambda_{J_l} \cdot \Lambda_{J_l'})$. By Lemma 6.2, $y_1 = G_{J_l}(x_1, \Lambda_{J_l})$ and $y_2 = G_{J_l'}(x_2, \Lambda_{J_l'})$. Furthermore,

$$\begin{aligned} & G_I(y_1 \cdot y_2, \Lambda_{I_1} \cdot \Lambda_{I_2}) \\ = \;& \varphi_J(x_1 \cdot x_2, \Lambda_{J_l} \cdot \Lambda_{J_l'} + \Lambda_{I_1} \cdot \Lambda_{I_2}) \\ = \;& \varphi_J(x_1 \cdot x_2, (\Lambda_{J_l} + \Lambda_{I_1}) \cdot (\Lambda_{J_l'} + \Lambda_{I_2})). \end{aligned}$$

By Lemma 6.2,

$$\varphi_J(x_1 \cdot x_2, (\Lambda_{J_l} + \Lambda_{I_1}) \cdot (\Lambda_{J_l'} + \Lambda_{I_2})) = (G_{J_l}(x_1, \Lambda_{J_l} + \Lambda_{I_1}), G_{J_l'}(x_2, \Lambda_{J_l'} + \Lambda_{I_2})).$$

Since $|p, r_l| < n$ and $|r_l, q| < n$, by the outer inductive hypothesis, G_{J_l}, $G_{J_l'}$, G_{I_1} and G_{I_2} satisfy (4.1). Thus

$$\begin{aligned} & (G_{J_l}(x_1, \Lambda_{J_l} + \Lambda_{I_1}), G_{J_l'}(x_2, \Lambda_{J_l'} + \Lambda_{I_2})) \\ = \;& (G_{I_1}(G_{J_l}(x_1, \Lambda_{J_l}), \Lambda_{I_1}), G_{I_2}(G_{J_l'}(x_2, \Lambda_{J_l'}), \Lambda_{I_2})) \\ = \;& (G_{I_1}(y_1, \Lambda_{I_1}), G_{I_2}(y_2, \Lambda_{I_2})). \end{aligned}$$

This completes the proof of the lemma. □

We have defined the desired G_I in $\mathrm{Im}\varphi_J$ for all I such that $\mathcal{M}_I \cap \mathrm{Im}\varphi_J \neq \emptyset$. Clearly, $(\mathcal{M}_I \cap \mathrm{Im}\varphi_J) \times [0, \epsilon)^{|I|}$ has a frame $\{\frac{\partial}{\partial \lambda_1}, \cdots, \frac{\partial}{\partial \lambda_{|I|}}\}$. Then

$$\{\mathcal{N}_1(I), \cdots, \mathcal{N}_{|I|}(I)\} = dG_I|_{\Lambda_J = 0} \cdot \left\{\frac{\partial}{\partial \lambda_1}, \cdots, \frac{\partial}{\partial \lambda_{|I|}}\right\}$$

serves a desired frame of $A((\mathcal{M}_I \cap \mathrm{Im}\varphi_J), \overline{\mathcal{M}(p,q)})$. Identify $\mathrm{Im}\varphi_J$ with $\mathcal{M}_J \times [0, \epsilon)^{|J|}$ via φ_J, give $\mathrm{Im}\varphi_J$ the product connection (See Definition 5.9 and the comment following it.). Again, $G_I(y, \Lambda_I) = \varphi_J(x, \Lambda_J + \Lambda_I)$, and $\Lambda_J + t\Lambda_I$ for $t \in [0,1]$ is a line segment in $[0, \epsilon)^{|J|}$. Then $G_I(y, t\Lambda_I)$ is a geodesic segment. Thus $G_I(y, \Lambda_I) = \exp(y, \sum_{i=1}^{|I|} \lambda_i \mathcal{N}_i(I))$ and this connection is the desired one.

Do the above construction for each J such that $|J| = n$. Clearly, $G_J = \varphi_J$ when $|J| = n$. Let $U_n = \bigcup_{|J|=n} \mathrm{Im} G_J$, then $U_n \supseteq X_n$. This completes the first step of the inner induction.

(II). The second step of the inner induction (the induction on U_k).

Suppose we have constructed $U_{k+1} = \bigcup_{|I_0| \geq k+1} \mathrm{Im} G_{I_0}$. Suppose, for all I, we have constructed $G_I|_{U_{k+1}}$, the frames on $\mathcal{M}_I \cap U_{k+1}$ and the connection on U_{k+1} which provides G_I via exponential maps. Moreover, $(\mathcal{M}_I \cap U_{k+1}) \times [0, \epsilon)^{|I|}$ has a product connection if we pull back the connection on U_{k+1} via G_I. We shall extend the above things to those on U_k.

The construction shares many details with the first step. The essential point is that the definition of $G_I|_{U_k}$ should be an extension of $G_I|_{U_{k+1}}$.

Let $U_{k+1}(\delta) = \bigcup_{|I| \geq k+1} G_I|_{U_{k+1}}(\mathcal{M}_I \times [0, \delta)^{|I|})$ for $\delta \in (0, \epsilon)$. It's an open set such that $X_{k+1} \subset U_{k+1}(\delta) \subset U_{k+1}$. Let $\overline{U_{k+1}(\delta)} = \bigcup_{|I| \geq k+1} G_I|_{U_{k+1}}(\mathcal{M}_I \times [0, \delta]^{|I|})$.

Lemma 6.5. *The set $\overline{U_{k+1}(\delta)}$ is closed.*

Proof. For each I_0 such that $|I_0| \geq k+1$, we have $\overline{\mathcal{M}_{I_0}} = \bigsqcup_{I_0 \preceq I} \mathcal{M}_I$ is compact. Moreover, $G_{I_0}|_{U_{k+1}} : \mathcal{M}_{I_0} \times [0,\epsilon)^{|I_0|} \to \overline{\mathcal{M}(p,q)}$ has been defined.

Define $\overline{G_{I_0}} : \overline{\mathcal{M}_{I_0}} \times [0,\epsilon)^{|I_0|} \to \overline{\mathcal{M}(p,q)}$ as $\overline{G_{I_0}}(x,\Lambda_{I_0}) = G_I|_{U_{k+1}}(x,\Lambda_{I_0,I})$ for $(x,\Lambda_{I_0}) \in \mathcal{M}_I \times [0,\epsilon)^{|I_0|}$. Since the maps $G_I|_{U_{k+1}}$ satisfy (4.1), we infer that $\overline{G_{I_0}}$ is well defined and is a smooth embedding.

Thus $\overline{U_{k+1}(\delta)} = \bigcup_{|I_0| \geq k+1} \overline{G_{I_0}}(\overline{\mathcal{M}_{I_0}} \times [0,\delta]^{|I_0|})$ is compact. □

As the first step, by Lemma 5.8, for each J such that $|J| = k$, there is a smooth embedding $\varphi_J : \mathcal{M}_J \times [0,\epsilon_0)^{|J|} \to \overline{\mathcal{M}(p,q)}$ satisfying the stratum condition. Thus $d\varphi_J \cdot \{\frac{\partial}{\partial\lambda_1}, \cdots, \frac{\partial}{\partial\lambda_{|J|}}\}$ is a frame satisfying the stratum condition (See Corollary 5.7). By the inner inductive hypothesis, $\mathcal{M}_J \cap U_{k+1}$ already has a frame $\{\mathcal{N}_1(J), \cdots, \mathcal{N}_{|J|}(J)\}$ satisfying the stratum condition. Both $N_i(J)$ and $d\varphi_J \frac{\partial}{\partial\lambda_i}$ represent nonzero elements in the same $A(\mathcal{M}_J,\mathcal{M}_I) \cong [0,+\infty)$ for some $I \prec J$ such that $|I| = |J|-1$. Thus, for all $\alpha(x) \geq 0$, $\{\alpha(x)N_i(J) + (1-\alpha(x))d\varphi_J \frac{\partial}{\partial\lambda_i} \mid i = 1,\cdots,n\}$ is also a frame satisfying the stratum condition. By Lemma 6.5 and the partition of unity,, there is a frame satisfying the stratum condition and coinciding with the old one in $U_{k+1}(\delta)$ for some $\delta > 0$. Also by the same reason, there is a connection in $U_{k+1} \cup \mathrm{Im}\varphi_J$ such that it coincides with the old one in $U_{k+1}(\delta)$. Then, by the above frame and connection, we can modify φ_J such that it coincides with $G_J|_{U_{k+1}}$ in $U_{k+1}(\delta)$. Since $\mathcal{M}_J - U_{k+1}(\delta) = \overline{\mathcal{M}_J} - U_{k+1}(\delta)$ is compact, and $G_J|_{U_{k+1}}$ is an embedding, by Lemma 6.5, we infer φ_J is an embedding defined on $\mathcal{M}_J \times [0,\epsilon_0)^{|J|}$ for some $\epsilon_0 \in (0,\delta]$. Just as the first step, we can modify φ_J furthermore such that it satisfies the conclusion of Lemma 6.2. Since originally φ_J and $G_J|_{U_{k+1}}$ coincide in $U_{k+1}(\delta)$ and $G_J|_{U_{k+1}}$ satisfies (4.2), the modification does not change $\varphi_J|_{U_{k+1}(\delta)}$. Thus the modified φ_J still coincides with $G_J|_{U_{k+1}}$ in $U_{k+1}(\delta)$.

The big difference between this step and the first step is as follows. In the first step, $\mathrm{Im}\varphi_J$ are pairwise disjoint for $|J| = n$. Thus there is no contradiction of the definition when G_I is defined in each $\mathrm{Im}\varphi_J$. Now it's impossible to make $\mathrm{Im}\varphi_J$ pairwise disjoint. We shall control their pair-wise intersections. Suppose $J_1 \neq J_2$ and $|J_1| = |J_2| = k$. Then $(\mathcal{M}_{J_1} - U_{k+1}(\delta)) \cap (\mathcal{M}_{J_2} - U_{k+1}(\delta)) \subseteq \mathcal{M}_{J_1} \cap \mathcal{M}_{J_2} = \emptyset$. Since $\mathcal{M}_{J_i} - U_{k+1}(\delta)$ is compact, shrink ϵ_0 if necessary, we have

$$\varphi_{J_1}\left((\mathcal{M}_{J_1} - U_{k+1}(\delta)) \times [0,\epsilon_0)^{|J_1|}\right) \cap \varphi_{J_2}\left((\mathcal{M}_{J_2} - U_{k+1}(\delta)) \times [0,\epsilon_0)^{|J_2|}\right) = \emptyset.$$

Since

$$\varphi_{J_i}\left((\mathcal{M}_{J_i} \cap U_{k+1}(\delta)) \times [0,\epsilon_0)^{|J_i|}\right) \subseteq U_{k+1}(\delta),$$

we get $\mathrm{Im}\varphi_{J_1} \cap \mathrm{Im}\varphi_{J_2} \subseteq U_{k+1}(\delta)$.

Now we define G_I in each $\mathrm{Im}\varphi_J$. We only need to consider I such that $I \preceq J$. For all $y \in \mathcal{M}_I \cap \mathrm{Im}\varphi_J$, $y = \varphi_J(x,\Lambda_J)$, define $\widetilde{G}_I(J)(y,\Lambda_I) = \varphi_J(x,\Lambda_J+\Lambda_I)$. Given $\varphi_J = G_J|_{U_{k+1}}$ in $U_{k+1}(\delta)$, similarly to the argument in the first step, we get $\widetilde{G}_I(J) = G_I|_{U_{k+1}}$ in $U_{k+1}(\delta)$. Since $\mathrm{Im}\varphi_{J_1} \cap \mathrm{Im}\varphi_{J_2} \subseteq U_{k+1}(\delta)$, $\widetilde{G}_I(J_1)$ coincides with $\widetilde{G}_I(J_2)$ in their common domains. Define $G_I|_{\mathrm{Im}\varphi_J} = \widetilde{G}_I(J)$. Then G_I is well defined on $U_{k+1}(\delta) \cup \bigcup_{|J|=k} \mathrm{Im}\varphi_J$ and it coincides with $G_I|_{U_{k+1}}$ in $U_{k+1}(\delta)$.

Similarly to the first step, the maps $G_I|_{\mathrm{Im}\varphi_J}$ satisfy (4.1) and (4.2).

Shrink U_{k+1} to be $U_{k+1}(\epsilon_0)$. Again, $G_J = \varphi_J$ when $|J| = k$. Let

$$U_k = U_{k+1} \cup \bigcup_{|J|=k} G_J(\mathcal{M}_J \times [0,\epsilon_0)^k).$$

The desired $G_I|_{U_k}$ is defined in the above. Shrink ϵ_I to be ϵ_0 for all I. Give frames to $\mathcal{M}_I \cap U_k$ as the first step. For $|J| = k$, give $\mathrm{Im}G_J$ the product connection via G_J. The old connection in U_{k+1} is the product connection. Thus the new connection in $\mathrm{Im}G_J$ coincides with the old one in U_{k+1}. This completes the second step of the inner induction.

(III). The completion of the second step of the outer induction (the induction on $|p, q|$).

For the fixed pair (p, q), the construction in U_k requires a shrink of ϵ_I for all I with head p and tail q. However, the inner induction stops in a finite number of steps. Eventually, we have $\epsilon_I > 0$ which are the same for all I with head p and tail q. And if $I_1 \cdot I_2 = I$, then $\epsilon_I \leq \epsilon_{I_i}$. Thus we have constructed the desired G_I for the pair (p, q) with length n. This completes the second step of the outer induction and also the proof of this theorem. □

7. A Byproduct

The argument for Theorem 4.4 already gives the following Proposition 7.1 which gives a compatible collar structure for an arbitrary compact manifold with faces.

Suppose L is a smooth manifold with faces. Suppose F_i $(i = 1, \cdots, n)$ are its faces such that $\bigcup_{i=1}^n F_i = \bigcup_{k>0} \partial^k L$. In other words, $\bigcup_{i=1}^n F_i$ is the full boundary of L. Suppose the interiors of F_i are pairwise disjoint.

Let $I = \{i_1, \cdots, i_k\}$ be a subset of $\{1, \cdots, n\}$. Define $|I| = k$. Define $F_I = \bigcap_{i \in I} F_i$. In particular, when $I = \emptyset$, define $F_\emptyset = L$. Then, by Lemma 5.3, F_I is either empty or an $n-k$ dimensional smoothly embedded submanifold with corners insider L. Denote the interior of F_I by F_I°.

Let $V_I = \prod_{i \in I}[0, +\infty)$ be a factor space of $[0, +\infty)^n$. In other words, V_I is the product of the ith coordinate spaces of $[0, +\infty)^n$ such that $i \in I$. In particular, $V_\emptyset$ consists of one point. Let $V_I(\epsilon) = \prod_{i \in I}[0, \epsilon)$.

Let $\Lambda_I = \{\lambda_{i_1}, \cdots, \lambda_{i_k}\} \in V_I$ represent the collaring parameter for F_I°. Suppose $J \subseteq I$. Define $\Lambda_I(I - J) \in V_I$ as

$$\Lambda_I(I-J)(i) = \begin{cases} 0 & i \in J, \\ \lambda_i & i \in I - J. \end{cases}$$

Define $\Lambda_{I,J} \in V_J$ as $\Lambda_{I,J}(i) = \lambda_i$ for $i \in J$.

Proposition 7.1. *Suppose L is compact. Then collaring maps $G_I : F_I^\circ \times V_I(1) \to L$ can be defined for all I such that $F_I^\circ \neq \emptyset$. These maps satisfy the following conditions:*

(1). They are smooth embeddings which satisfy the following stratum condition. If $J \subseteq I = \{i_1, \cdots, i_k\}$, $\Lambda_I = \{\lambda_{i_1}, \cdots, \lambda_{i_k}\} \in V_I(1)$, and $\lambda_i = 0$ if and only if $i \in J$, then $G_I(x, \Lambda_I) \in F_J^\circ$ for all $x \in F_I^\circ$. In particular, $G_\emptyset : F_\emptyset^\circ = \partial^0 L \to L$ is the inclusion.

(2). They satisfy the following compatibility. If $J \subseteq I$ and $\lambda_i > 0$ when $i \notin J$, then, for all $x \in F_I^\circ$, we have

$$G_I(x, \Lambda_I) = G_J(G_I(x, \Lambda_I(I-J)), \Lambda_{I,J}).$$

The assumption of Proposition 7.1 is more general than that of Theorem 4.4 in some sense. However, this proof is actually even easier than that one because we only deal with one manifold with faces. It only requires that (4.1) is true in a more general setting. We don't need any more the arguments related to (4.2) such as

Lemmas 6.2 and 6.4. Instead of a double induction, it suffices to repeat the inner induction in the proof of Theorem 4.4. Since there are only finitely many set I, we can find $\epsilon > 0$ such that $\epsilon_I = \epsilon$ for all I. By a scaling of parameter, we get $\epsilon = 1$, which finishes the proof.

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